QUASI-COMMUTING FAMILIES OF QUANTUM MINORS

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In [7] a combinatorial criterion for quasi-commutativity is established for pairs of quantum Plücker coordinates in the quantized coordinate algebra $\mathbb{C}_q[\mathcal{F}]$ of the flag variety of type A. This paper attempts to generalize these results by producing necessary and sufficient conditions for pairs of quantums minors in the quantized coordinate algebra $\mathbb{C}_q[Mat_{k\times m}]$ to quasi-commute. In addition we study the combinatorics of maximal (by inclusion) families of pairwise quasi-commuting quantum minors and pose relevant conjectures.

1. Introduction

Let $\mathbb{C}_q[Mat_{k\times m}]$ be the q-deformation of the coordinate ring of the space of $k\times m$ complex matrices where $k \le m$. This is the $\mathbb{C}(q)$-algebra with unity generated by indeterminates $x_{i,j}$ for $i \in [1\dots k]$ and $j \in [1\dots m]$ subject to the Faddeev-Reshetikhin-Takhtadzhyan relations [2]:

$$\begin{array}{ll} x_{s,t}x_{i,j} = q\ x_{i,j}x_{s,t} & \text{if either } s>i \text{ and } t=j \\ & \textbf{or} \quad s=i \text{ and } t>j \\ x_{s,t}x_{i,j} = x_{i,j}x_{s,t} & \text{if } s>i \text{ and } t<j \\ x_{s,t}x_{i,j} = x_{i,j}x_{s,t} + (q-q^{-1})\ x_{i,t}x_{s,j} & \text{if } s>i \text{ and } t>j \end{array}$$

In this paper we shall be concerned with a special family of elements $\Delta_{I,J} \in \mathbb{C}_q[Mat_{k\times m}]$ indexed by pairs of non-empty subsets I and J of $[1\dots k]$ and $[1\dots m]$ respectively with $|I| = |J| = l$. They are defined by:

$$\Delta_{I,J} := \sum_{\sigma\in S_l} (-q)^{-l(\sigma)} x_{i_1,j_{\sigma(1)}} \cdots x_{i_l,j_{\sigma(l)}} ,$$

where $I = \{i_1 < \cdots < i_l\}$, $J = \{j_1 < \cdots < j_l\}$, and $l(\sigma)$ is the length of the l-permutation σ. The element $\Delta_{I,J}$ is the q-deformation of the classical determinant and for this reason we call the $\Delta_{I,J}$'s quantum minors.

Definition 1. *Two quantums minors $\Delta_{A,B}$ and $\Delta_{C,D}$ quasi-commute if $\Delta_{C,D}\Delta_{A,B}$ $= q^c\ \Delta_{A,B}\Delta_{C,D}$ for some integer c. The integer c is uniquely determined by $\Delta_{A,B}$ and $\Delta_{C,D}$ and we will denote its value by the symbol $c\big(\Delta_{A,B} \mid \Delta_{C,D}\big)$. Note that $c\big(\Delta_{C,D} \mid \Delta_{A,B}\big) = -c\big(\Delta_{A,B} \mid \Delta_{C,D}\big)$ for any quasi-commuting pair.*

We can now state the central problems we will address is this paper, namely:

Problem 1. *Find necessary and sufficient conditions for two quantum minors $\Delta_{A,B}$ and $\Delta_{C,D}$ to quasi-commute. In addition, explicitly compute $c\big(\Delta_{A,B} \mid \Delta_{C,D}\big)$ in terms of A, B, C, and D.*

Problem 2. *Find a combinatorial mechanism which will describe and produce all maximal (by inclusion) families of pairwise quasi-commuting quantum minors.*

Problems 1 and 2 are motivated by the study of dual canonical bases for quantum groups of type A. It is conjectured in [1], and partially proved in [8], that products of quasi-commuting quantum minors constitute a part of the dual canonical basis for the quantum group $\mathbb{C}_q[GL(n,\mathbb{C})]$. Problem 2 is also motivated by the study of total positivity as described in [3] and [4].

Problem 1 is resolved using techniques developed in [7]. Ostensibly Problem 1 is more general than its counterpart in [7] which only addresses the quantum flag variety. Nevertheless we demonstrate in this paper that Problem 1 can be reduced to a special case of the problem treated in [7] - namely the problem of determining when two quantum Plücker coordinates of the corresponding quantum Grassmannian quasi-commute. The criterion for quasi-commutativity is described in terms of the notion of "weak separability" as put forth in [7].

Definition 2. *Given two subsets I and J of $[1\dots n]$ we write $I \prec J$ if $i < j$ for all $i \in I$ and all $j \in J$. We say I and J are weakly separated if at least one of the following two conditions holds:*

1. *$|I| \geq |J|$ and $J-I$ can be partitioned into a disjoint union $J-I = J' \sqcup J''$ so that $J' \prec I-J \prec J''$.*
2. *$|J| \geq |I|$ and $I-J$ can be partitioned into a disjoint union $I-J = I' \sqcup I''$ so that $I' \prec J-I \prec I''$.*

We associate to any pair of subsets $A \subset [1\dots k]$ and $B \subset [1\dots m]$ of equal size the subset $S(A,B) \subset [1\dots k+m]$ of size k defined as follows:

$$S(A,B) = \Big\{\ b+k\ \Big\|\ b \in B\ \Big\} \sqcup [1\dots k] - w_0(A)$$

where w_0 is the order reversing permutation of $[1\dots k]$. Problem 1 is settled by the following two Theorems:

Theorem 1. *The quantum minors $\Delta_{A,B}$ and $\Delta_{C,D}$ in $\mathbb{C}_q[Mat_{k,m}]$ quasi-commute if and only if $S(A,B)$ and $S(C,D)$ are weakly separated subsets of $[1\dots m+k]$.*

Theorem 2. *Suppose $I = S(A,B)$ and $J = S(C,D)$ are weakly separated subsets of $[1 \dots m+k]$ satisfying case 1 in Definition 2. Then*

$$c\big(\Delta_{A,B} \mid \Delta_{C,D}\big) = |J''| - |J'| + |A| - |C|.$$

In proving Theorems 1 and 2 we use a quantum analogue of the well known embedding of $Mat_{k\times m}$ as an affine chart in the Grassmannian $\mathbb{G}_{k,k+m}$; this embedding sends a $k \times m$ matrix $(x_{i,j})$ to the row space of the $k \times (k+m)$ matrix

$$\begin{pmatrix} 0 & & & & 1 & x_{1,1} & \cdot & \cdot & \cdot & \cdot & x_{1,m} \\ & & & -1 & & \cdot & & & & & \cdot \\ & & \cdot & & & \cdot & & & & & \cdot \\ & \cdot & & & & \cdot & & & & & \cdot \\ \cdot & & & & & \cdot & & & & & \cdot \\ (-1)^{k-1} & & & & 0 & x_{k,1} & \cdot & \cdot & \cdot & \cdot & x_{k,m} \end{pmatrix}$$

The corresponding quantum analogue is an embedding of $\mathbb{C}_q[Mat_{k\times m}]$ into the quantized coordinate ring $\mathbb{C}_q[\mathbb{G}_{k,k+m}]$ - the so called quantum Grassmannian as defined in [10]. This embedding allows us to reduce questions about quantum minors to corresponding questions about quantum Plücker coordinates.

Theorem 1 implies that $\mathcal{C} = \{\Delta_{A_1,B_1}, \dots, \Delta_{A_s,B_s}\}$ is a maximal collection of pairwise quasi-commuting quantum minors in $\mathbb{C}_q[Mat_{k,m}]$ if and only if $\big\{S(A_1,B_1), \dots, S(A_s,B_s)\big\} \sqcup \big\{[1\dots k]\big\}$ is a maximal collection of pairwise weakly separated k-subsets of $[1 \dots k+m]$. This identification is a central component in our attempt to resolve Problem 2. Theorem 1.3 of [7] asserts that the size of any maximal collection of pairwise weakly separated k-subsets of $[1\dots n]$ is sharply bounded by $k(n-k)+1$. Setting $n = k+m$ we obtain:

Proposition 1. *The size of any maximal collection of pairwise quasi-commuting quantum minors in $\mathbb{C}_q[Mat_{k\times m}]$ is sharply bounded by km.*

In [7] the following *purity* property is conjectured: all maximal collections of pairwise weakly separated subsets (not neccessarily k-subsets) of $[1\dots n]$ have size $\binom{n+1}{2}+1$. The analogue of this purity conjecture for k-subsets is given by:

Conjecture 1 (Purity)**.** *All maximal collections of pairwise weakly separated k-subsets of $[1\dots n]$ have size $k(n-k)+1$. Equivalently, all maximal collections of pairwise quasi-commuting quantum minors in $\mathbb{C}_q[Mat_{k\times m}]$ have size km.*

In Sections 5 and 6 we prove this assertion for the cases $k=2$ and $k=3$ respectively.

In Section 3 we expose a new feature specific to the quantum Grassmannian: quasi-commutativity of the quantum Plücker coordinates in $\mathbb{C}_q[\mathbb{G}_{k,n}]$ is preserved under the natural action of the dihedral group D_n. More precisely, we show that the

natural D_n-action on k-subsets of $[1 \dots n]$ preserves weak separbility. We do not know of an analogue of this action for the full quantum flag variety. Let $\mathrm{W}(k, n)$ be the set of all maximal collections of pairwise weakly separated k-subsets in $[1 \dots n]$. The induced D_n-action on $\mathrm{W}(k, n)$ is instrumental in proving several assertions in this paper.

For a set I and elements x and y let I_{xy} denote $I \cup \{x, y\}$. The set $\mathrm{W}(k, n)$ possesses the following interesting structure.

Theorem 3. *Let $\mathcal{C}$ be a maximal collection of pairwise weakly separated k-subsets of $[1 \dots n]$. Suppose that I_{ij}, I_{it}, I_{js}, $I_{st} \in \mathcal{C}$ for some $i < s < j < t$ and for some $I \subset [1 \dots n] - \{i, j, s, t\}$ with $|I| = k - 2$. Then $\mathcal{C}$ contains either I_{ij} or I_{st} and not both. Moreover, the transformation*

$$\mathcal{C} \longmapsto \begin{cases} \mathcal{C} - \{I_{ij}\} \sqcup \{I_{st}\} & \text{if } \; I_{ij} \in \mathcal{C} \\ \mathcal{C} - \{I_{st}\} \sqcup \{I_{ij}\} & \text{if } \; I_{st} \in \mathcal{C} \end{cases} \tag{1}$$

preserves weak separability and maximality.

This transformation is an analogue of the Yang-Baxter "flip" introduced in [7]; here we refer to these transformations as $(2, 4)$-moves due to the fact that they originate on $\mathbb{C}_q[\mathbb{G}_{2,4}]$.

Conjecture 2 (Transitivity)**.** *Let $\mathcal{C}$ and $\mathcal{B}$ be any collections in $\mathrm{W}(k, n)$. Then there is a sequence of $(2, 4)$-moves transforming $\mathcal{C}$ into $\mathcal{B}$.*

If true the conjecture effectively settles Problem 2. In addition it provides a method to obtain all collections in $\mathrm{W}(k, n)$: simply propagate a given maximal collection by all possible $(2, 4)$-moves. In Section 3 we explain why the validity of Conjecture 2 implies the validity of Conjecture 1. In Sections 5 and 6 we prove this Conjecture 2 for the cases $k = 2$ and $k = 3$. In Section 8 we explore applications of this conjecture to total positivity.

In Section 4 we describe certain maximal collections in $\mathrm{W}(k, n)$ arrising from *double wiring arrangements*. In Section 7 we present a construction that recursively generates all collections in $\mathrm{W}(3, n)$ by lifting collections from $\mathrm{W}(3, n-1)$. In principle this construction should provide a method to compute the size of $\mathrm{W}(3, n)$.

2. The Quantum Grassmannian and Proofs of Theorems 1 and 2

Definition 3. *The quantum Grassmannian $\mathbb{C}_q[\mathbb{G}_{k,n}]$, as defined in [10], is the $\mathbb{C}(q)$-algebra with unity generated by all quantum Plücker coordinates Δ^K where K is a k-subset of $[1 \dots n]$ subject to the relations:*

$$\sum_{i \in I - J} (-q)^{\mathrm{inv}(i,I) - \mathrm{inv}(i,J)} \; \Delta^{I - \{i\}} \; \Delta^{J \sqcup \{i\}} \; = \; 0$$

for any $(k+1)$-subset I and $(k-1)$-subset J. Here $\mathrm{inv}(i, X)$ *is the number of $x \in X$ such that $i > x$.*

Proposition 2 (Quantum Stieffel-Plücker Correspondence)**.** *There exists a unique $\mathbb{C}(q)$-algebra embedding $\varphi : \mathbb{C}_q[Mat_{k\times m}] \longrightarrow \mathbb{C}_q[\mathbb{G}_{k,k+m}]$ such that*

$$\Delta_{I,J} \longmapsto q^{\binom{l}{2}}\ \Delta^{l-1}\ \Delta^{S(I,J)}$$

where $l = |I| = |J|$ and $\Delta = \Delta^{[1\ldots k]}$.

Proof. The proof that the Faddeev-Reshetikhin-Takhtadzhyan relations are preserved under the correspondence $x_{i,j} \longmapsto \Delta^{S(\{i\},\{j\})}$ and that $\Delta_{I,J}$ is sent to $q^{\binom{l}{2}}\ \Delta^{l-1}\ \Delta^{S(I,J)}$ is a simple modification of the proof of the quantum analogue of Bazin's theorem presented in Theorem 3.8 of [6].

The classical analogue of φ, obtained by specializing q to 1, is easily seen to be injective. This taken together with Theorem 3.5(c) of [10] and the fact that the monomials consisting of products of lexicographically ordered generators $x_{i,j}$ form a basis for $\mathbb{C}_q[Mat_{k\times m}]$ over $\mathbb{C}(q)$ proves injectivity of φ.

□

It is well known that $\Delta^{[1\ldots k]}$ is quasi-central. Thus Proposition 2 tells us that two quantum minors $\Delta_{A,B}$ and $\Delta_{C,D}$ will quasi-commute exactly when the corresponding quantum Plücker coordinates $\Delta^{S(A,B)}$ and $\Delta^{S(C,D)}$ quasi-commute. In turn, the conditions for two quantum Plücker coordinates to quasi-commute are explained by the following proposition of [7]:

Proposition 3. *Two quantum Plücker coordinates Δ^I and Δ^J in $\mathbb{C}_q[\mathbb{G}_{k,n}]$ quasi-commute if and only if I and J are weakly separated. If I and J satisfy case 1 of Definition 2 then $c(\Delta^I \mid \Delta^J) = |J''| - |J'|$.*

Theorem 1 now follows from Propositions 2 and 3. Theorem 2 also follows from Propositions 2 and 3 along with the fact that $c(\Delta^{|A|-1} \mid \Delta^{S(C,D)}) = |C|(|A|-1)$ and $c(\ \Delta^{S(A,B)} \mid \Delta^{|C|-1}) = |A|(1-|C|)$.

3. Proof of Theorem 3

It is convenient to visualize a k-subset of $[1\ldots n]$ as a subpolygon of the regular polygon with n vertices labeled counter-clockwise by the indices $[1\ldots n]$. Represent the dihedral group D_n as the group of symmetries of the n-gon. Clearly D_n acts on the set of k-subsets of $[1\ldots n]$ under this realization.

Proposition 4. *If two k-subsets I and J of $[1\ldots n]$ are weakly separated then $g(I)$ and $g(J)$ are weakly separated for any $g \in D_n$.*

Proof. In [7] it is shown that I and J are weakly separated precisely when, after interchanging I and J if neccessary, either:

a) $|I| < |J|$ and there do not exist three indices $a < b < c$ such that $I \cap \{a, b, c\} = \{b\}$ and $J \cap \{a, b, c\} = \{a, c\}$ **or**

b) $|I| = |J|$ and there do not exist four indices $a < b < c < d$ such that $I \cap \{a, b, c, d\} = \{a, c\}$ and $J \cap \{a, b, c, d\} = \{b, d\}$

Part b) above indicates that two k-subsets I and J are weakly separated precisely, when viewed as subpolygons, no diagonal of the subpolygon I disjoint from J *crosses* a diagonal of J disjoint from I. This property is clearly preserved under any dihedral symmetry of the n-gon.

$\square$

A k-subset I is called *boundary* if it consists of k consecutive indices of the n-gon; i.e. any k-subset of the form $g([1 \dots k])$ for $g \in D_n$. Since $[1 \dots k]$ is weakly separated with every k-subset it follows that the set of all k-boundary subsets is common to every maximal collection of pairwise weakly separated k-subsets.

Proof of Theorem 3:

To prove the first part of the theorem notice that since I_{ij} and I_{st} are not weakly separated it is clear that both can not be in $\mathcal{C}$. So we need only demonstrate that one of them is present in $\mathcal{C}$. Given a k-subset J of $[1 \dots n]$ such that J is weakly separated from I_{is}, I_{sj}, I_{jt}, I_{it} and different from I_{ij} and I_{st} we need to show that J is weakly separated from both I_{ij} and I_{st}.

Proposition 4 shows that we may reduce the proof to the case of $t = n$ after suitably translating the collection $\mathcal{C}$ by the *dihedral action.* Assume that $t = n$. Let $J^- = J - \{n\}$. Since $|J| = k$ and J is different from I_{ij} and I_{st}, it follows that J^- is different from both I_{ij} and I_s. By Lemma 3.2 of [7], J^- is weakly separated from I_{is}, I_{sj}, I_j, I_i. By Lemma 5.2 of [7], it follows that J^- is weakly separated from both I_{ij} and I_s and, after an easy application of part b) above, that J is weakly separated from both I_{ij} and I_{sn}, as claimed.

The above argument also shows that the transformation (1) preserves weak separability and maximality, thus concluding the proof of Theorem 3. $\square$

Returning to Conjecture 2, notice that if it is true and if we can find a collection $\mathcal{A}$ in $\mathrm{W}(k, n)$ for which $|\mathcal{A}| = k(n-k) + 1$ then Conjecture 1 will follow. One can easily verify that the collection $\mathcal{A} = \mathcal{A}_n$ whose non-boundary sets are

$$\left\{ \ [1 \dots i] \sqcup [j \dots k + j - i - 1] \ \middle\| \ 1 \leq i < k \text{ and } i + 1 < j < n + i - k \ \right\}$$

has the desired properties.

4. Wiring Arrangements

In [7] a recursive procedure is described through which all maximal families of pairwise weakly separated subsets (not neccessarily k-subsets) of $[1\ldots n]$ are obtained. In principle this recursion can be restricted to produce all families in $\mathrm{W}(k,n)$. Nevertheless, the process is not very practical. In this section we explore a non-recursive combinatorial device which parametrizes a large portion of the collections in $\mathrm{W}(k,n)$. This device is a modification of a construction in [3].

Recall that the symmetric group S_n is generated by the simple reflections $s_i = (i,i+1)$ satisfying the Coxeter relations. A reduced word for an element $g \in S_n$ is sequence of indices $i_1,\ldots i_l$ such that $g = s_{i_1}\cdots s_{i_l}$ with l minimal. For the group $S_k \times S_m$ we will use the indices $[\overline{1},\ldots,\overline{k-1}]$ to label the simple reflections corresponding to the S_k component and the indices $[1\ldots m-1]$ to label the simple reflections for the S_m component. Under this convention a reduced word for an element $(u,v) \in S_k \times S_m$ can be identified with a shuffle of a reduced word for u, written with indices in $[\overline{1},\ldots,\overline{k-1}]$, and a reduced word for v written with indices $[1\ldots m-1]$.

Let $w_0^{(k)}$ and $w_0^{(m)}$ denote the longest elements in S_k and S_m respectively. We say a reduced word for $(w_0^{(k)}, w_0^{(m)}) \in S_k \times S_m$ is *optimal* if the associated reduced word for $w_0^{(m)} \in S_m$ has a total of only $\binom{m-k}{2}$ occurrences of the indices $[k+1,\ldots m-1]$. Given an opitimal reduced word $\mathbf{i}$ of $(w_0^{(k)}, w_0^{(m)}) \in S_k \times S_m$ we will manufacture a maximal collection $\mathcal{C}(\mathbf{i})$ of pairwise quasi-commuting quantum minors. This collection is obtained by means of the *double wiring arrangement* $\mathrm{Arr}(\mathbf{i})$ attached to $\mathbf{i}$, as introduced in [3].

Recall first the definition of a *single wiring arrangement* attached to a reduced word. It is easiest to understand this definition with an example. Consider the reduced word 1231 of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \in S_4$. The corresponding single wiring arrangement is:

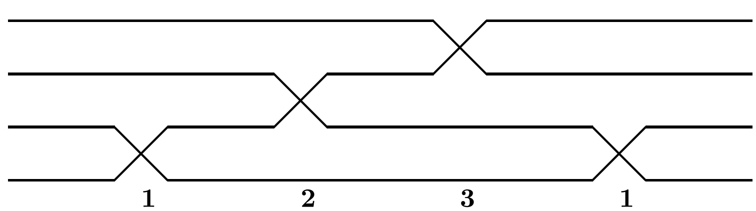

Figure 1. Single wiring arrangement

We associate a crossing at the ith level (counting from the bottom up) for each i in the reduced word. To obtain the double wiring arrangement for $(u,v) \in S_k \times S_m$ we superimpose the single wiring arrangements for the reduced words of u and v respectively aligning them closely in the vertical direction (starting at the bottom)

and intertwining their respective crossings as dictated by the shuffle. To distinguish the two wiring arrangements we colour the diagram for u red. For example, the double wiring arrangement corresponding to the reduced word

$$\mathbf{i} = 2\ \overline{1}\ 1\ 2\ 3\ \overline{2}\ 2\ 1\ 4\ \overline{1}\ 3\ 2\ 1 \quad \text{for} \quad (w_0^{(3)}, w_0^{(5)}) \in S_3 \times S_5 \quad \text{is:}$$

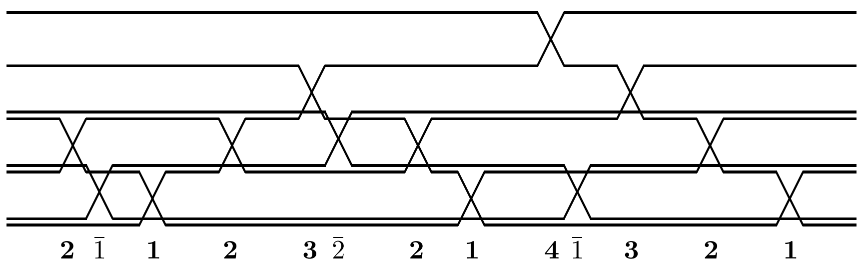

FIGURE 2. Double wiring arrangement

To obtain the collection $\mathcal{C}(\mathbf{i})$ label the black wires 1 through m bottom-up at the left hand side of the arrangement and label the red wires 1 through k bottom-up at the right hand side of the arrangement. Label each chamber C in the first k strips of the arrangement with $I(C)$ - the set of labels of red lines passing beneath the chamber - and $J(C)$ - the set of black line labels passing beneath the chamber. For example the above double wiring arrangement is labeled

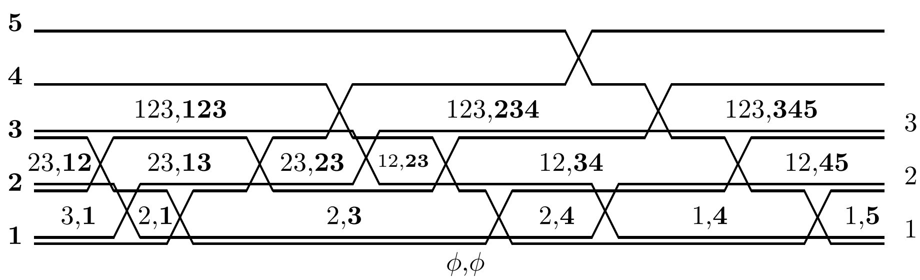

FIGURE 3. Labeled arrangement

Let $\mathcal{C}(\mathbf{i}) = \left\{ \Delta_{I(C),J(C)} \;\middle\|\; C \text{ a chamber of } \mathrm{Arr}(\mathbf{i}) \text{ of level} \leq k \right\}$.

Lemma 1. *Let* $\mathbf{i}$ *be an optimal reduced word for* $(w_0^{(k)}, w_0^{(m)}) \in S_k \times S_m$. *Then the size of* $\mathcal{C}(\mathbf{i})$ *is* km.

Proof. Given $\mathbf{i}$, the number of chambers in the first k strips of the corresponding double wiring arrangement is equal to the number of red and black crossings in the first k strips plus k - corresponding to the k far right chambers . The number of black (respectively red) crossings in the first k strips in turn is given by the number of simple reflections j (respectively $\overline{j}$) occurring in the reduced word $\mathbf{i}$ with $1 \leq j \leq k$. The number of $\overline{j}$ in $\mathbf{i}$ with $1 \leq j \leq k$ is $\binom{k}{2}$. The number of of j in $\mathbf{i}$ with $1 \leq j \leq k$ is $\binom{m}{2} - \#\left\{ j \text{ occurring in } \mathbf{i} \;\middle\|\; k+1 \leq j \leq m-1 \right\}$; if $\mathbf{i}$ is *optimal*

this will be $\binom{m}{2} - \binom{m-k}{2}$. Consequently the number of chambers occurring in the first k strips of the double wiring arrangement for $\mathbf{i}$ optimal (or equivalently the size of $\mathcal{C}(\mathbf{i})$) is:

$$\binom{k}{2} + \binom{m}{2} - \binom{m-k}{2} + k = mk$$

$\square$

Proposition 5. *If $\mathbf{i}$ is an optimal reduced word for $(w_0^{(k)}, w_0^{(m)}) \in S_k \times S_m$ then $\mathcal{C}(\mathbf{i})$ is a maximal collection of pairwise quasi-commuting quantum minors in $\mathbb{C}_q[Mat_{k\times m}]$. Moreover, given $\Delta_{A,B}$ and $\Delta_{I,J}$ in $\mathcal{C}(\mathbf{i})$ either*

$$A - I \prec I - A \quad \textit{and} \quad J - B \prec B - J \quad \textbf{or} \tag{2}$$

$$I - A \prec A - I \quad \textit{and} \quad B - J \prec J - B \tag{3}$$

Proof. Take any quantum minors $\Delta_{A,B}$ and $\Delta_{I,J}$ in $\mathcal{C}(\mathbf{i})$. Lemma 4.1 of [7] proves that if X and Y are chamber sets of a single wiring arrangement then either $X - Y \prec Y - X$ or $Y - X \prec X - Y$. This, taken together with the fact that the single wiring arrangements for the S_k and S_m components of $\mathbf{i}$ are oppositely labeled, proves the second part of the proposition.

To prove that $\Delta_{A,B}$ and $\Delta_{I,J}$ quasi-commute we must show that $S(A,B)$ and $S(I,J)$ are weakly separated. We may assume, after exchanging A with I and B with J if neccessary, that $A - I \prec I - A$ and $J - B \prec B - J$. This in turn is equivalent to

$$\Big(S(A,B) - S(I,J)\Big) \cap [1\dots k] \prec S(I,J) - S(A,B) \prec \Big(S(A,B) - S(I,J)\Big) - [1\dots k]$$

which demonstrates that $S(A,B)$ and $S(I,J)$ are weakly separated. The fact that $\mathcal{C}(\mathbf{i})$ is maximal follows from Lemma 1 and Proposition 1.

$\square$

It is possible to prove the converse of Proposition 5, namely: If $\mathcal{C}$ is a collection of quantum minors $\Delta_{A,B}$ whose indices pairwise satisfy either condition 2 or 3, and if $\mathcal{C}$ is maximal with respect to this property, then $\mathcal{C}$ is of the form $\mathcal{C}(\mathbf{i})$ for some optimal reduced word $\mathbf{i}$.

Given an optimal reduced word $\mathbf{i}$ the following collection is in $\mathrm{W}(k, k+m)$:

$$\Big\{ \; S\Big(I(C), J(C)\Big) \;\Big\|\; C \text{ a chamber of } \mathrm{Arr}(\mathbf{i}) \text{ of level } \leq k \;\Big\} \sqcup \Big\{[1\dots k]\Big\}$$

In the case of $\mathrm{W}(3,6)$ all collections are obtained via double wiring arrangements. There are 34 in total and they are explicitly described in [3] and [4]. Every maximal family in $\mathrm{W}(3,6)$ is dihedrally equivalent to one of the following five collections (we omit boundary sets):

$$\Big\{ \{124\},\{125\},\{134\},\{145\} \Big\} \quad \Big\{ \{124\},\{125\},\{145\},\{245\} \Big\}$$

$$\Big\{ \{124\},\{134\},\{145\},\{146\} \Big\} \quad \Big\{ \{125\},\{134\},\{135\},\{145\} \Big\}$$

$$\Big\{ \{135\},\{136\},\{145\},\{235\} \Big\}$$

In general it is not the case that every maximal collection in $\mathrm{W}(k,n)$ corresponds to some double wiring arrangement, even after dihedral translation. This is evidenced already in the case of $\mathbb{C}_q[\mathbb{G}_{2,n}]$. In Section 5 we shall demonstrate such a maximal collection.

5. The case of $\mathbb{C}_q[\mathbb{G}_{2,n}]$

We identify the 2-subsets of $[1\dots n]$ with chords inscribed in a regular n-gon. Clearly two 2-subsets of $[1\dots n]$ are weakly separated if and only if the corresponding chords do not cross in the interior of the polygon. Under this identification collections $\mathcal{C} \in \mathrm{W}(2,n)$ correspond to maximal collections of non-crossing chords - i.e. triangulations of an n-gon.

Theorem 4 (Transitivity)**.** *Let $\mathcal{C},\mathcal{B} \in \mathrm{W}(2,n)$. Then there is a sequence of $(2,4)$-moves transforming $\mathcal{C}$ into $\mathcal{B}$.*

Proof. This theorem follows from the well known fact that the any two triangulations are connected by a series of chord exchanges where the diagonal chord of an inscribed quadralateral is "flipped" to its crossing pair. The diagonal "flips" correspond to $(2,4)$-moves. □

Corollary 1 (Purity)**.** *Let $\mathcal{C} \in \mathrm{W}(2,n)$. Then $|C| = 2(n-2)+1$.*

Proof. Immediate corollary of Theorem 4.

□

Since $\mathrm{W}(2,n)$ is identified with the set of triangulations of an n-gon it follows that $|\mathrm{W}(2,n)|$ is the Catalan number $\frac{1}{n-1}\binom{2n-4}{n-2}$. For $k>2$ the size of $\mathrm{W}(k,n)$ is not known.

In [5] it is shown that the coordinate ring $\mathbb{C}[\mathbb{G}_{2\times n}]$ has a basis consisting of all monomials of Plücker coordinates whose indices are pairwise weakly separated. Using the quantum short Plücker relation given by

$$\Delta^{Iij}\,\Delta^{Ist} \;=\; q\,\Delta^{Iis}\,\Delta^{Ijt} \;+\; q^{-1}\,\Delta^{Iit}\,\Delta^{Isj}$$

for $i < s < j < t$ as a straightening rule, we obtain the following quantum analogue of this result:

Proposition 6. *The set of all monomials consisting of lexicographically ordered pairwise quasi-commuting quantum Plücker coordinates is a basis for* $\mathbb{C}_q[\mathbb{G}_{2,n}]$.

Using Proposition 5 and the identification of maximal collections in $\mathrm{W}(2,n)$ with triangulations of an n-gon we can characterize those maximal collections which can be parametrized, up to the dihedral action, by double wiring arrangements. Given $\mathcal{C} \in \mathrm{W}(2,n)$ there exists $g \in D_n$ for which $g \cdot \mathcal{C}$ is parametrized by a double wiring arrangement if and only if there exists an external edge of the polygon (i.e. a boundary 2-set) such that for any other external edge there is **no** chord in the associated triangulation, which separates both the edges and is disjoint from both. The following collection in $\mathrm{W}(2,9)$, represented as a triangulation, is an example of a collection which is not parametrized, up to the dihedral action, by a double wiring arrangement:

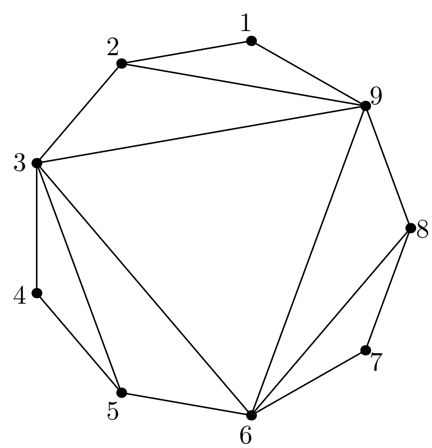

FIGURE 4. Non-Parametrized $\mathrm{W}(2,9)$ collection

6. The case of $\mathbb{C}_q[\mathbb{G}_{3,n}]$

In this section we prove the Transitivity and Purity Conjectures for $k = 3$.

Theorem 5 (Transitivity)**.** *Let* $\mathcal{C}, \mathcal{B} \in \mathrm{W}(3,n)$. *Then there is a sequence of* $(2,4)$*-moves transforming* $\mathcal{C}$ *into* $\mathcal{B}$.

Corollary 2 (Purity)**.** *Let* $\mathcal{C} \in \mathrm{W}(3,n)$ *then* $|\mathcal{C}| = 3(n-3)+1$.

Proof of Transitivity:

The essential strategy is to show that any collection $\mathcal{C} \in \mathrm{W}(3,n)$ can be reduced by a sequence of $(2,4)$-moves to the "base" collection $\mathcal{A}_n$ whose non-boundary 3-sets are

$$\left\{ \{1, s, s+1\} \;\middle\|\; 2 < s < n-1 \right\} \bigsqcup \left\{ \{1, 2, s\} \;\middle\|\; 3 < s < n \right\}$$

We first prove that whenever a collection $\mathcal{C}$ can be $(2,4)$-reduced to $\mathcal{A}_n$ then so can any of its dihedral translations $g \cdot \mathcal{C}$ for $g \in D_n$. In Lemma 3 we then show that any maximal collection can be translated dihedrally to a maximal collection containing the 3-set $\{1, n-2, n-1\}$. We conclude the proof by showing that any such collection can be reduced by a sequence of $(2,4)$-moves to the collection $\mathcal{A}_n$.

Lemma 2. *Let $\mathcal{C} \in \mathrm{W}(3,n)$. If $\mathcal{C}$ can be reduced by a sequence of $(2,4)$-moves to $\mathcal{A}_n$ then so can the collection $g \cdot \mathcal{C}$ for any $g \in D_n$.*

Proof. Since the D_n-action preserves $(2,4)$-moves it is enough to verify this assertion in the case where $\mathcal{C} = \mathcal{A}_n$.

Proceed by induction on n. For $n \leq 4$ the statement is evident. Assume $n > 4$. It is enough to verify the claim for the group elements ρ_n and σ_n, which generate D_n, given by

$$\rho_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} \quad \sigma_n = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots \\ 2 & 1 & n & n-1 & n-2 & \cdots \end{pmatrix}$$

This follows from the observation that if $g \cdot \mathcal{C}$ can be reduced by a sequence of $(2,4)$-moves to $\mathcal{B}$ then $hg \cdot \mathcal{C}$ can be reduced to $h \cdot \mathcal{B}$.

The collection $\sigma_n \cdot \mathcal{A}_n$ contains the 3-sets $\{1, 2, n-1\}$, $\{2, n-2, n-1\}$, $\{n-2, n-1, n\}$, $\{1, n-1, n\}$, and $\{2, n-1, n\}$. Applying the $(2,4)$-move which replaces $\{2, n-1, n\}$ with $\{1, n-2, n-1\}$ we obtain $\sigma_{n-1} \cdot \mathcal{A}_{n-1} \sqcup \left\{ \{1, 2, n\}, \{1, n-1, n\}, \{n-2, n-1, n\} \right\}$. By induction $\sigma_{n-1} \cdot \mathcal{A}_{n-1}$ can be reduced by a sequence of $(2,4)$-moves to $\mathcal{A}_{n-1}$. Thus $\sigma_n \cdot \mathcal{A}_n$ can be reduced to $\mathcal{A}_{n-1} \sqcup \left\{ \{1, 2, n\}, \{1, n-1, n\}, \{n-2, n-1, n\} \right\} = \mathcal{A}_n$.

To deal with ρ_n, notice that $\rho_n \cdot \mathcal{A}_n$ contains the 3-sets $\{1, 2, n\}$, $\{1, 2, 3\}$, $\{2, 3, n-1\}$, $\{2, n-1, n\}$, and $\{2, 3, n\}$. We apply the $(2,4)$-move which replaces $\{2, 3, n\}$ with $\{1, 2, n-1\}$. This new collection contains the 3-sets $\{1, n-1, n\}$, $\{1, 2, n-1\}$, $\{2, n-2, n-1\}$, $\{n-2, n-1, n\}$, and $\{2, n-1, n\}$. We may apply the $(2,4)$-move which replaces $\{2, n-1, n\}$ with $\{1, n-1, n-2\}$. The resulting collection is exactly $\rho_{n-1} \cdot \mathcal{A}_{n-1} \sqcup \left\{ \{1, 2, n\}, \{1, n-1, n\}, \{n-2, n-1, n\} \right\}$. By the induction hypothesis $\rho_{n-1} \cdot \mathcal{A}_{n-1}$ can be reduced by a sequence of $(2,4)$-moves to $\mathcal{A}_{n-1}$. Consequenlty $\rho_n \cdot \mathcal{A}_n$ can be reduced to $\mathcal{A}_{n-1} \sqcup \left\{ \{1, 2, n\}, \{1, n-1, n\}, \{n-2, n-1, n\} \right\} = \mathcal{A}_n$.

$\square$

Lemma 3. *Given $\mathcal{C} \in \mathrm{W}(3,n)$ there exists $g \in D_n$ such that $g \cdot \mathcal{C}$ contains the 3-set $\{1, n-2, n-1\}$.*

Proof. For a 3-subset I of $[1 \dots n]$ define the diameter of I to be the minimal cardinality of a boundary k-subset of $[1 \dots n]$ that contains I. Thus the boundary 3-subsets are precisely those of diameter 3. Let us call 3-subsets of diameter 4 *almost boundary* subsets. It suffices to prove that every maximal collection $\mathcal{C}$ contains an almost boundary subset.

Assume by contradiction that $\mathcal{C}$ does not contain an almost boundary 3-subset. We make the following easy observation:

Remark 1. *Let a, b, c, and d be four consecutive vertices in $[1 \dots n]$; then the 3-subsets that are not weakly separated with an almost boundary subset $\{a, c, d\}$ are precisely the non-boundary 3-subsets containing b but not a.*

Therefore our assumption and maximality of $\mathcal{C}$ imply that for every two consecutive vertices a and b in $[1 \dots n]$, there is a non-boundary 3-subset in $\mathcal{C}$ which contains b but not a.

Choose a non-boundary 3-subset $\{a, c, d\}$ in $\mathcal{C}$ of minimal possible diameter. Without loss of generality, we can assume that a boundary subset of minimal cardinality that contains $\{a, c, d\}$ has a and d as its endpoints; let us denote this boundary subset by $[a, d]$. We can also assume that c is not a neighbor of a. Let b be the neighbor of a in $[a, d]$. Consider a 3-subset I in $\mathcal{C}$ such that I contains b but not a. Since I is weakly separated from $\{a, c, d\}$ it must be contained in $[b, d] = [a, d] - \{a\}$. But then I has smaller diameter than $\{a, c, d\}$ which contradicts our choice of $\{a, c, d\}$. This proves the claim and hence the lemma as well.

$\square$

For any collection $\mathcal{C} \in \mathrm{W}(3, n)$ we define its *height* $H(\mathcal{C})$ to be the number of non-boundary 3-sets containing n. An immediate consequence of Remark 1 is that $H(\mathcal{C}) = 0$ if and only if both $\{1, 2, n-1\}$ and $\{1, n-2, n-1\}$ are in $\mathcal{C}$.

Lemma 4. *Let $\mathcal{C} \in \mathrm{W}(3, n)$ with $\{1, n-2, n-1\} \in \mathcal{C}$. Then $\mathcal{C}$ can be reduced by a sequence of $(2, 4)$-moves to a collection of height $H = 0$.*

Proof. We proceed by induction on the height. If $H(\mathcal{C}) = 0$ then we are already done. Assume inductively that the assertion is true for collections of height $H = k \geq 0$ and let $\mathcal{C}$ be a collection of height $H(\mathcal{C}) = k + 1$. We need the following:

Lemma 5. *Let $\mathcal{C} \in \mathrm{W}(3, n)$ and suppose that $\{1, n-2, n-1\} \in \mathcal{C}$. Then there exists a unique index $b > 1$ such that both $\{1, b, n-1\}$ and $\{1, b, n\}$ are in $\mathcal{C}$. We call b the pinch point over n and $n-1$.*

Proof. Let b be the maximal index with the property that $\{1, b, n\} \in \mathcal{C}$. Suppose, by contradiction, that $\{1, b, n-1\} \notin \mathcal{C}$. By maximality of $\mathcal{C}$ this means there exists a non-boundary set $I \in \mathcal{C}$ which is not weakly separated with $\{1, b, n-1\}$. Therefore there exist indices $s, t \in I$ such that one of the following holds:

1. $1 < s < b < t < n-1$
2. $1 < s < b$ and $t = n$
3. $b < s < n-1$ and $t = n$

Case 1: Since I and $\{1, b, n\}$ are weakly separated it follows that $b \in I$. But then I will be weakly separated with $\{1, b, n-1\}$.

Case 2: Since $\{1, n-2, n-1\} \in \mathcal{C}$ and since I is a non-boundary set containing n it follows that $1 \in I$. But then I will be weakly separated with $\{1, b, n-1\}$.

Case 3: Once again it must be the case that $1 \in I$. So $I = \{1, s, n\}$ where $b < s$ violating the maximalitly of b.

Hence $\{1, b, n-1\} \in \mathcal{C}$. Suppose there was another pinch point $b' \neq b$. Either $b' < b$ or $b' > b$. If $b' < b$ then $\{1, b', n-1\}$ will not be weakly separated from $\{1, b, n\}$. If $b' > b$ then $\{1, b', n\}$ will not be weakly separated from $\{1, b, n-1\}$. Both possibilities violate that fact that $\mathcal{C}$ consists of only pairwise weakly separated 3-sets. Uniqueness follows.

□

Lemma 6. *Let $\mathcal{C} \in \mathrm{W}(3, n)$ and assume $\{1, n-2, n-1\} \in \mathcal{C}$. Let b be the pinch point over n and $n-1$. Assume in addition that $b > 2$. Then there exists a with $1 < a < b$ such that both $\{1, a, b\}$ and $\{1, a, n\}$ are in $\mathcal{C}$.*

Proof. Consider the set of all x with the property that $x < b$ and $\{1, x, n\} \in \mathcal{C}$. This set is clearly non-empty since $2 < b$ and $\{1, 2, n\} \in \mathcal{C}$. Let a be the maximal index with this property. Suppose $\{1, a, b\} \notin \mathcal{C}$. Then there exists $I \in \mathcal{C}$ with $s, t \in I$ such that one of the following holds:

1. $1 < s < a < t < b$
2. $1 < s < a < b < t$
3. $a < s < b < t$

Case 1: Since $\{1, a, n\} \in \mathcal{C}$ it follows that I and $\{1, a, n\}$ must be weakly separated. The only way this can happen is that $a \in I$. But then I and $\{1, a, b\}$ will be weakly separated.

Case 2: Since I and $\{1, a, n\}$ are weakly separated it must be the case that $t = n$. Since $\{1, n-2, n-1\} \in \mathcal{C}$ it follows that I and $\{1, n-2, n-1\}$ are weakly separated. The only way this can be resolved is that $1 \in I$. But then I and $\{1, a, b\}$ are weakly separated.

Case 3: Either $t = n$ or not. Suppose $t \neq n$. Since $\{1, b, n\} \in \mathcal{C}$, and hence weakly separated from I, it follows that $b \in I$ in which case I and $\{1, a, b\}$ will be weakly separated. Thus $t = n$. Since $\{1, b, n-1\} \in \mathcal{C}$ we know that I and $\{1, b, n-1\}$ are weakly separated. The only way this can happen is that $1 \in I$ and hence $I = \{1, s, n\}$. But this violates the maximality of a since $a < s < b$.

Thus $\{1, a, b\}$ and $\{1, a, n\}$ are in $\mathcal{C}$ as required.

□

Returning to Lemma 4, let b be the pinch point of $\mathcal{C}$ - i.e. the unique index b such that both $\{1, b, n-1\}$ and $\{1, b, n\}$ are in $\mathcal{C}$. If $b = 2$ it follows that $\{1, 2, n-1\} \in \mathcal{C}$.

This, taken together with the fact that $\{1, n-2, n-1\} \in \mathcal{C}$, violates the hypothesis that $H(\mathcal{C}) > 0$. Therefore $b > 2$.

Since $b > 2$ Lemma 6 implies that there exists a with $1 < a < b$ such that both $\{1, a, b\}$ and $\{1, a, n\}$ are in $\mathcal{C}$. Thus $\mathcal{C}$ contains $\{1, a, b\}$, $\{1, a, n\}$, $\{1, b, n-1\}$, $\{1, b, n\}$, and $\{1, n-1, n\}$. The associated $(2,4)$-move for this quintuple replaces $\{1, b, n\}$ with $\{1, a, n-1\}$. Let $\mathcal{B}$ be the resulting collection. Notice that $\mathcal{B}$ contains $\{1, n-2, n-1\}$ and that $H(\mathcal{B}) = H(\mathcal{C}) - 1 = k$. By induction $\mathcal{B}$ can be further reduced by a sequence of $(2,4)$-moves into a collection of height $H = 0$. Concantenating this $(2,4)$-reduction with the $(2,4)$-move transforming $\mathcal{C}$ to $\mathcal{B}$ we obtain the desired reduction for $\mathcal{C}$.

□

Now we are ready to finish the proof of Transitivity. Let $\mathcal{C} \in \mathrm{W}(3, n)$. By Lemma 3 there is $g \in D_n$ such that $g \cdot \mathcal{C}$ contains the 3-set $\{1, n-2, n-1\}$. By Lemma 4 the collection $g \cdot \mathcal{C}$ can be reduced by a sequence of $(2,4)$-moves to a collection $\mathcal{B}$ with height $H(\mathcal{B}) = 0$. The collection $\mathcal{B} - \Big\{ \ \{1, 2, n\}, \{1, n-1, n\}, \{n-2, n-1, n\} \ \Big\}$ is in $\mathrm{W}(3, n-1)$ and by induction on n we can assume that it can be reduced by a sequence of $(2,4)$-moves to $\mathcal{A}_{n-1}$. Equivalently $\mathcal{B}$ can be reduced by a sequence of $(2,4)$-moves to $\mathcal{A}_n$. Consequently $g \cdot \mathcal{C}$ can be reduced to $\mathcal{A}_n$ and applying Lemma 2 we conclude that $\mathcal{C}$ can be reduced to $\mathcal{A}_n$ as required.

7. Reduction

In this section we present a recursive procedure to generate collections in $\mathrm{W}(3, n)$.

Given a 3-subset I of $[1 \dots n]$, we define

$$I' = \begin{cases} I \sqcup \{n-1\} - \{n\} & \text{if } n \in I \text{ and } n-1 \notin I \\ \phi & \text{if } n \in I \text{ and } n-1 \in I \\ I & \text{if } n \notin I \end{cases}$$

For $\mathcal{C} \in \mathrm{W}(3, n)$ let $\mathcal{C}' = \{ \ I' \mid I \in \mathcal{C} \ \}$, and define $F_{\mathcal{C}}$ to be the set of indices $b \in [2 \dots n-1]$ with $\{1, b, n\} \in \mathcal{C}$ such that $\{1, b\} - \{s, t\} \prec \{s, t\} - \{1, b\}$ whenever $\{s, t, n\} \in \mathcal{C}$ for $1 < s < t$. If $\mathcal{C}$ contains $\{1, n-2, n-1\}$, let $b_{\mathcal{C}}$ be the pinch point of $\mathcal{C}$ (see Lemma 5), that is, the unique index such that both $\{1, b_{\mathcal{C}}, n-1\}$ and $\{1, b_{\mathcal{C}}, n\}$ are in $\mathcal{C}$.

Theorem 6 (Reduction)**.** *Let $n \geq 4$. The mapping $\mathcal{C} \longmapsto \big(\mathcal{C}', b_{\mathcal{C}}\big)$ defines a bijection between collections in $\mathrm{W}(3, n)$ containing $\{1, n-2, n-1\}$ and the set*

$$\Big\{ \ \big(\mathcal{B}, b\big) \in \mathrm{W}(3, n-1) \times [2 \dots n-2] \ \Big\| \ b \in F_{\mathcal{B}} \ \Big\}$$

The inverse bijection sends a pair $\big(\mathcal{B}, b\big)$ *to the collection* $\hat{\mathcal{B}}_b := \left\{ \ I_b \ \big\| \ I \in \mathcal{B} \ \right\} \sqcup \Big\{\{1, b, n-1\}, \{1, n-1, n\}, \{n-2, n-1, n\}\Big\}$ *where*

$$I_b = \begin{cases} I - \{n-1\} \sqcup \{n\} & \text{if } n-1 \in I \text{ and } I - \{1, b, n-1\} \prec \ \{1, b\} - I \\ I & \text{otherwise} \end{cases}$$

Since by Lemma 3, every collection in $\mathrm{W}(3, n)$ is dihedrally equivalent to one containing the near boundary subset $\{1, n-2, n-1\}$, it follows from Theorem 6 that all collections in $\mathrm{W}(3, n)$ can be obtained by first lifting collections in $\mathrm{W}(3, n-1)$ by the inverse of the reduction procedure and then translating them suitably by the dihedral action.

Proof of Reduction Theorem:

The following lemma shows that the mapping $\mathcal{C} \longmapsto \big(\mathcal{C}', b_c\big)$ is well defined.

Lemma 7. *Let* $\mathcal{C} \in \mathrm{W}(3, n)$. *Then* $\mathcal{C}' \in \mathrm{W}(3, n-1)$, *and* $b_c \in F_{\mathcal{C}'}$.

Proof. Momentary consideration reveals that $\mathcal{C}'$ consists of pairwise weakly separated 3-subsets of $[1 \ldots n-1]$. In virtue of Corollary 2 we know that $\mathcal{C}'$ will be maximal if and only if $|\mathcal{C}'| = 3(n-4)+1$. Since $\{1, n-2, n-1\} \in \mathcal{C}$ it follows that if $I \in \mathcal{C}$ and $I' = \phi$ then either $I = \{1, n-1, n\}$ or $I = \{n-2, n-1, n\}$. Consequently $|\mathcal{C}'| \leq |\mathcal{C}| - 2$. For $I, J \in \mathcal{C}$ if $I' = J'$ then either $I = J$ or else there exists $b \in [2 \ldots n-2]$ such that, after interchanging I and J if neccessary, $I = \{1, b, n-1\}$ and $J = \{1, b, n\}$. By Lemma 5, b is unique. Hence $|\mathcal{C}'| = |\mathcal{C}| - 3 = 3(n-4)+1$ as required. The inclusion $b_c \in F_{\mathcal{C}'}$ is also clear from the definitions.

□

To prove that the inverse correspondence is well defined, we need to show that $\hat{\mathcal{B}}_b \in \mathrm{W}(3, n)$ and $\{1, n-2, n-1\} \in \hat{\mathcal{B}}_b$ for any $\mathcal{B} \in \mathrm{W}(3, n-1)$ and $b \in F_{\mathcal{B}}$. Simple consideration shows that all 3-subsets in $\hat{\mathcal{B}}_b$ are weakly separated because $b \in F_{\mathcal{B}}$. Since $\mathcal{B}$ is maximal we know by Corollary 2 that $|\mathcal{B}| = 3(n-4)+1$ and thus $|\hat{\mathcal{B}}_b| = |\mathcal{B}| + 3 = 3(n-3)+1$. Corollary 2 implies that $\hat{\mathcal{B}}_b \in \mathrm{W}(3, n)$. Notice also that $\{1, n-2, n-1\} \in \hat{\mathcal{B}}_b$ since $b \leq n-2$.

It remains to show that the mappings $\mathcal{C} \longmapsto \big(\mathcal{C}', b_c\big)$ and $\big(\mathcal{B}, b\big) \longmapsto \hat{\mathcal{B}}_b$ are inverse to each other. First suppose that $\mathcal{C} = \hat{\mathcal{B}}_b$. Since both $\{1, b, n\}$ and $\{1, b, n-1\}$ are in $\hat{\mathcal{B}}_b$, the desired equality $\big(\mathcal{C}', b_c\big) = \big(\mathcal{B}, b\big)$ follows from Lemma 5. Finally, the equality $\widehat{\mathcal{C}'_b} = \mathcal{C}$ for $b = b_c$ is clear from the definitions. □

Example: Let $\mathcal{C}$ be the collection in $\mathrm{W}(3, 6)$ whose non-boundary 3-sets are

$$\Big\{\{136\}, \{146\}, \{236\}, \{346\}\Big\}$$

Here $F_c = \{2, 3\}$. Notice that $4 \notin F_c$ because $\{1, 4\} - \{23\} \not\prec \{2, 3\} - \{1, 4\}$. The index 5 is not present for the same reason. The two possible lifts of $\mathcal{C}$ (omitting boundaries) are:

$$\hat{\mathcal{C}}_2 = \Big\{\{126\}, \{136\}, \{146\}, \{156\}, \{236\}, \{346\}\Big\}$$

$$\hat{\mathcal{C}}_3 = \Big\{\{137\}, \{136\}, \{146\}, \{156\}, \{236\}, \{346\}\Big\}$$

8. Positivity

Let $\mathbb{G}_{k,n}(\mathbb{C})$ be the Grassmannian of k-subspaces in $\mathbb{C}^n$. Recall that any k-subspace in $\mathbb{G}_{k,n}(\mathbb{C})$ can be represented by a $k \times n$ matrix whose rows span the k-subspace. The Plücker coordinates are the maximal minors of this $k \times n$ matrix. We say a point $p \in \mathbb{G}_{k,n}(\mathbb{C})$ is positive if it can be represented by a $k \times n$ matrix whose Plücker coordinates $\Delta^I(p)$ are positive real numbers.

Definition 4. *Let $\mathcal{C}$ be a collection of k-subsets of $[1 \dots n]$. We say that $\mathcal{C}$ is a positivity test if $p \in \mathbb{G}_{k,n}(\mathbb{C})$ is positive if and only if all $\Delta^I(p)$ are real and positive for each $I \in \mathcal{C}$.*

In [7] it is conjectured that maximal families of pairwise weakly separated subsets (not necessarily k-subsets) of $[1 \dots n]$ give rise to positivity tests for the flag variety of type A_n. The analogue of this result for the Grassmannian $\mathbb{G}_{k,n}(\mathbb{C})$ is:

Theorem 7. *Let $k = 2$ or $k = 3$. If $\mathcal{C}$ is a maximal collection of pairwise weakly separated k-subsets of $[1 \dots n]$ then the associated collection of Plücker coordinates $\{\ \Delta^I \mid I \in \mathcal{C}\ \}$ is a positivity test.*

Proof. Let $\mathcal{C} \in \mathrm{W}(k, n)$ and suppose that all $\Delta^I(p)$ are real and positive for $I \in \mathcal{C}$. We need to show that all other Plücker coordinates $\Delta^J(p)$ are real and positive. Take any $J \notin \mathcal{C}$. Take any maximal collection $\mathcal{B}$ containing J. Since k is either 2 or 3 we know that Conjecture 2 holds and thus $\mathcal{C}$ and $\mathcal{B}$ are connected by a sequence of $(2, 4)$-moves.

Claim: Suppose $\mathcal{A}$ is in $\mathrm{W}(k, n)$ and is a positivity test. Let $\mathcal{B}$ be in $\mathrm{W}(k, n)$ and assume that $\mathcal{B}$ is obtained from $\mathcal{A}$ by a single $(2, 4)$-move. Then $\mathcal{B}$ is a positivity test.

Indeed, since $\mathcal{A}$ and $\mathcal{B}$ differ by a single $(2, 4)$-move there exist $i < s < j < t$ and I, where I is empty if $k = 2$ and $|I| = 1$ if $k = 3$, such that I_{is}, I_{sj}, I_{jt}, and I_{it} are in both $\mathcal{A}$ and $\mathcal{B}$ and such that, without loss of generality, $\mathcal{B}$ is obtained from $\mathcal{A}$ by replacing I_{ij} with I_{st}. The fact that $\mathcal{B}$ is a positivity test is an immediate consequence of the short Plücker relation

$$\Delta^{Iij}\ \Delta^{Ist}\ =\ \Delta^{Iis}\ \Delta^{Ijt}\ +\ \Delta^{Iit}\ \Delta^{Isj}$$

Let l be the minimal number of $(2, 4)$-moves required to join $\mathcal{B}$ and $\mathcal{C}$. To prove the theorem proceed by induction on l and use the claim.

□

A positivity test $\mathcal{C}$ is **minimal** if it has no proper subset which is also a positivity test. We conjecture that $\mathcal{C}$ is a minimal positivity test for $\mathbb{G}_{k,n}(\mathbb{C})$ if and only if $\mathcal{C}$ is in $\mathrm{W}(k,n)$. In addition, A. Zelevinsky and S. Fomin conjecture that collections $\mathcal{C}$ in $\mathrm{W}(k,n)$ have the property that any Plücker coordinate Δ^J can be uniquely expressed as a positive Laurent polynomial in the Plücker coordinates Δ^I for $I \in \mathcal{C}$. The author intends to investigate these issues related to positivity in a forthcoming article.

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